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One Problem, Two Contexts

Danielle L. Gigger
Brigham Young University - Provo

ONE PROBLEM, TWO CONTEXTS

by

Danielle Leigh Gigger

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Danielle Leigh Gigger

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date	Charles N. Walter, Chair
Date	Robert Speiser
Date	Steven R. Williams

BRIGHAM YOUNG UNIVERSITY

As chair of the candidate's graduate committee, I have read the thesis of Danielle Leigh Gigger in its final form and have found that (1) its format, citations and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

Date

Charles N. Walter
Chair, Graduate Committee

Accepted for the Department

Date

Charles N. Walter
Graduate Coordinator

Accepted for the College

Date

Tom Sederberg
Associate Dean, College of Physical and Mathematical Sciences

ABSTRACT

ONE PROBLEM, TWO CONTEXTS

Danielle Leigh Gigger

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In this study, a group of students were presented with two mathematically isomorphic problems but in radically different contexts. Analysis of their thinking and reasoning as they worked to solve and explain each problem demonstrates that the thinking and reasoning that emerged in each problem responded to clear purposes that the problems elicited in these students. The first problem was posed in a context that relied on experience and intuition rather than a formal mathematical description. The second problem was posed in a formal, set-theoretic context. While the analysis offered here reveals similarities in the students' final reasoning in the two contexts, it brings to light major differences between the purposes, choices, and reasoning in both contexts.

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Chapter I: Introduction and Background

Proof has been a research topic of interest in mathematics education for several decades. This research has included investigations into the aspects of proving and the role of proof in the development of mathematical thinking (Galbraith, 1981; Fischbein & Kedem, 1982). Research has also turned to the pedagogical aspects of proof, including topics such as preservice teachers' views of proof and students' transitions to formal proof (Martin & Harel, 1989; Hanna, 1990; Moore, 1994).

Considering the pedagogical aspects of proof, many have found that the role of proof in the classroom is often limited to logical verification. However, Hoyles (1997) states that if this is the only meaning students build in regard to proof, then students will likely encounter many conceptual difficulties in constructing proofs. Hanna (1990) asserts that if the goal of mathematical understanding is to be reached, explanatory proofs must be developed and used in a classroom. Reaffirming Hanna's ideas, Hoyles adds that rather than focusing only on verification, school proofs should offer students insight into why a given statement is true, illuminating the mathematical structures of the ideas. Indeed, many others have come to describe proof as a learning tool that offers both conviction *and* understanding to a particular mathematical assertion (Hersh, 1993; Weber, 2005).

Researchers conclude that for students to access proofs that offer both conviction and understanding, the social dimension must also be considered. Hanna (1991) discusses social interaction as a means of determining and accepting a convincing argument. Others add that students can gain conviction of truth as they are given opportunities to test and refine their conjectures as well as to present their generalizations and evidence for their ideas (Bell, 1976; Mason, Burton, & Stacey, 1982; Cockcroft, 1982). These perspectives lead to a view of proof as a problem solving setting in which students engage in social interaction, testing and refining conjectures as they work to build arguments that offer conviction as well as understanding of mathematical ideas.

Although, as indicated above, considerable research has been focused on proof, especially on proof as a problem solving activity, there still remains much to be understood. For example, Francisco and Maher (2005) state that, "more research is needed that can help provide a 'coherent explanatory frame' as to how problem solving and mathematical thinking fit together" (p. 362). Viewing problem solving as a setting in which proof takes place, the gap seems to lie in understanding how mathematical thinking and reasoning play out in a problem solving setting. The National Council of Teachers of Mathematics (NCTM) (2000) gives direction as to where we can look to fill this gap. They state, "Reasoning and proof cannot simply be taught in a single unit on logic" (p. 56). Rather, mathematical reasoning "must be developed through consistent use in many contexts" (p. 56). Thus, research that explores mathematical thinking and reasoning in more than one context can offer insight into how that thinking plays a part in a problem solving setting and in the construction of proofs.

This study concentrates on the thinking and reasoning of a group of students as they work to solve and explain two mathematically isomorphic problems that are presented in radically different contexts[1]. In particular, this study demonstrates that the thinking and reasoning that emerge in each of the two problems responds to clear

[1] The first problem was adapted by the instructor from a problem listed in D'Angelo and West (1997). The second problem was created by the instructor as a formalized version of the first.

purposes that each of the two problems elicited in these students. For this reason, their emergent thinking and reasoning will be referred to as their purposeful thinking and reasoning. The first problem was posed in a context that relied on experience and intuition to describe the problem situation rather than a formal mathematical description. Eleven weeks later, the second problem was described in a formal, set-theoretic context. While the analysis offered here reveals striking similarities in the students' final reasoning in the two contexts, it brings to light major differences between the purposeful thinking and reasoning in the first context and the purposeful thinking and reasoning in the second context.

Students' purposes, perhaps drawn from different contexts, could offer teachers insight into students' thinking and reasoning that emerge from those purposes. Thus, understanding these purposes could better prepare teachers to understand their students' work as those students develop and use problems in different contexts. Consider the use of "story" or "real-world" problems in a lesson. Often these problems are introduced after a general concept has been "taught" in order to illustrate the applicability of the general concept in real life situations. However, some teachers, on the contrary, use these problems to introduce a concept, encouraging students to discover the concept from real life situations. It is clear that the real life context of the problem will influence the students' purposes, choices, and reasoning, but *how*? And will that reasoning align with the goals of the lesson? Now consider the use of purely symbolic problems. Similar questions arise: With what purpose do students approach these problems? *How* will the context help the students to develop reasoning aligned with the goals of the lesson as they reason through these problems? By focusing on the students' purposes and emergent reasoning in both contexts, teachers can begin to answer these questions.

Chapter II: Theoretical Perspective

To carefully describe and analyze the differences and similarities in the students' purposeful thinking and reasoning requires interweaving several perspectives: proof as problem solving, communication of meaning through representation, representation as presentation to oneself or others, representation as an anchor for reasoning, and the evolving nature of representations.

Proof is often described as a problem-solving activity. Weber (2005) states that when proofs are constructed, they can be viewed as problem-solving tasks in which valid, logical arguments are built. Indeed, by viewing proof from a problem-solving perspective, the mathematical knowledge, insights, and reasoning of students can be observed.

Freeman (2000), a neurobiologist, offers insight into students' reasoning in problem-solving settings. He states:

> "Meanings exist only in brains, because each meaning expresses the entire history and experience of an individual…It is the limited sharing of meanings between brains for social purposes that requires reciprocal exchanges of representations, each presentation by a transmitting brain inducing the construction of a new meaning in the receiving brain" (p. 93).

Although Freeman believes that representations are always external, Maher (2002) asserts that the mental images that students build in mathematics are used in building representations of the ideas involved in their mathematical problem solving. Through public discourse, certain features of the internal, cognitive representations of students are made available. As these students explain and justify their ideas, they encounter new experiences that drive them to revisit and modify their existing representations.

In this study, a representation is defined as a presentation in accordance with the work of Speiser, Walter, and Maher (2003). Representations are presentations either to "oneself, as part of an ongoing thought…or to others, as part of an emerging discourse" (p. 2). Therefore, a representation includes the examples, "graphs, diagrams, written symbols, gestures, or specific language use" of the students (p. 2).

Scholars have argued that representations provide a foundation from which mathematical reasoning can build, allowing for successful proof production in a problem solving setting. Weber (2005) describes successful proof productions as those in which students develop and refine their informal representations and gain conviction and understanding by reasoning from their refined representations. Lester and Kehle (2003) also place representations at the center of problem solving. They state:

> Successful problem solving in mathematics involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve the tension or ambiguity (i.e., lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity (p. 510).

Speiser, Walter and Sullivan (2006) state that as the students in their study began their explorations, they were led "to the construction of one or more presentations of the problem situation, then to solutions to be justified through reasoning based on the way the given presentations have been structured" (manuscript in preparation).

Although each perspective above highlights the importance of representations in problem solving and proof production, this study combines these perspectives to afford a much more detailed look at the students' reasoning as well as the purposes that drive that reasoning. The analysis presented here might be thought of as a rope whose strands are the various perspectives above. Rather than examining each strand individually, all of the strands will be woven together, strengthening the analysis of how these students worked in each context. In particular, this holistic view shows how the purposes, choices, and reasoning, as well as the representations they develop, are influenced by the context of the problem.

The guiding questions of this study are as follows:

1. *How do the students' purposes and choices affect their use of particular representations in each context?* This question addresses the effects of context on the students' purposes and choices[2] and in turn on the representations that they use.
2. *What do those representations tell us about their reasoning in each context?* A closer examination of each representation will show how context affects their reasoning.
3. *How do their representations and reasoning compare between the two contexts?* This comparison focuses on how context affects the students' reasoning and the purposes that motivate that reasoning.

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[2] See Walter and Gerson (in press) for a look at students' purposeful choices.

Chapter III: Relevant Literature

This chapter focuses on past research concerning students' cognitive structures in building meaning as they begin to construct proofs, their use of and reasoning from representations, and the use of key ideas and warrants in building proofs. Literature in these areas shed light on the purposes and choices of the students in this study in each context as they begin to construct proofs, using key ideas and warrants in their construction. Literature on representations offers access to their purposes, choices, and reasoning as they construct proofs in a problem-solving setting.

The Concept-Understanding Scheme

Moore (1994) developed what he calls *The Concept-Understanding Scheme* building from a concept scheme developed by Vinner and others (Dreyfus, 1990; Tall and Vinner, 1981; Vinner, 1983; Vinner and Dreyfus, 1989). In accordance with the work of Vinner and others, Moore defines the *concept definition* as "the definition of a mathematical concept" and the *concept image* as "the cognitive structure in an individual's mind associated with the concept" (p. 252). This concept image includes all of the pictures that one develops in his/her mind in relation to that concept. Moore adds one more aspect to this concept scheme, *concept usage*. This refers to how one works with a concept whether it be doing proofs or creating or using examples.

This *Concept-Understanding Scheme* was useful to Moore (1994) in explaining the difficulties that students encountered with proofs. One particular insight into student difficulties that his scheme offered was when students have difficulty starting a proof. The main problem that students encountered was that they could not understand or use the language and notation. This resulted from students' lack of an intuitive understanding of the concepts and an inability to use their concept images to write a proof. When students could not understand or use the language and notation, they also failed to be able to generate and use examples. Each of these individual problems affected the students' abilities to start a proof. In this research, the two contexts in which the students problem solve offer a strong contrast of intuitive understanding.

Representations

Robert B. Davis (1984) stated, "Representations are fundamental to mathematical thought…Hence, a major question for mathematics education is to explore the *representations* which beginners, or experts, build in their own minds" (p. 78). This study strongly emphasizes representations, relying on Speiser, Walter, and Maher's (2003) definition of a representation as a presentation. Speiser, Walter and Sullivan (2006) investigate the reasoning that is drawn from the representations that students build and how that reasoning supports the justification of their solutions. Indeed the representations that students use in problem solving can become the foundation to their reasoning and proof. Maher and Kiczek (2000) found that in one student's efforts to find and justify a solution, he created a representation, modified it over time with his classmates, and used that representation to build an argument for that solution as well as to make connections between the various mathematical situations that he had explored.

Building Proofs

Sowder (1985) described problem solving as “what it is you do when you don’t know what to do” (p. 141). Davis (1985) stated that the formulation of problems and the solving of problems go hand in hand. Indeed, according to Cifarelli and Cai (2005), there are “varying degrees of sense making, problem posing, and problem solving” (p. 322).

Raman (2003) discusses how *key ideas* can offer insight into how people view proof. Raman defines a key idea as an informal idea that can be mapped to a formal proof. By connecting the public and private domains, key ideas offer both conviction and understanding by showing “*why* a particular claim is true” (p. 323). Raman explains that students need to think about key ideas in order to view mathematical proof from a mature standpoint. She offers the following example:

> Let’s see, an even function. There is only one thing about it, and that is its graph is reflected across the axis. Yeah, and you can be quite convinced that it is true by looking at the picture. If you said enough words about the picture, you’d have a proof (p. 323).

This example illustrates the use of a key idea, the symmetry of an even function, to provide understanding of the claim as well as conviction once the idea is translated into more formal notation.

Weber and Alcock (2005) found that when mathematicians review a proof, they do not worry so much about whether a claim is true, but rather they look to see whether that claim is warranted. According to Weber and Alcock, a claim was warranted when “there was a legitimate mathematical reason for asserting that the conclusion of the implication was a consequence of its antecedent” (p. 35). Toulmin’s (1969) argumentation model follows along these same lines. Toulmin’s model has three essential features: the *data*, the *conclusion*, and the *warrant*. The assertions or claims that one makes in an argument is what Toulmin refers to as the *conclusion*. The evidence that is put forth to support the conclusion is what he refers to as the *data*. The justification for why the "data necessitates the conclusion" is the *warrant* (Alcock & Weber, 2005, p. 35, Toulmin, 1969). These three features can become important features of proof production in a problem solving setting.

Chapter IV: Methodology

Background

The participants of this study were students at a major western private university enrolled in the Mathematics course titled *Foundations of Mathematics*. This course was what has come to be known as a transition course for mathematics students. It was designed to move students from their procedure-based experience in mathematics to a rigorous, formal mathematics. The class this study focused on consisted of twenty-four students arranged in four working groups of six students each. Students in this classroom were assigned by the instructor to their groups at the beginning of the term and remained in those groups for the entire semester. The research presented here focused on one of these groups of six students as they worked on two specific tasks. Students in the focus group – four females and two males – were undergraduates who had declared majors or minors in mathematics or mathematics education. They were in their first or second year of their respective majors. The focus group of this study was chosen based on their abilities to work cooperatively and to articulate their ideas. Although there were six students in this focus group, the data presented here will be drawn from the work of five of these students.

Setting

According to Brousseau (1999), a didactical contract is established in each classroom that determines in a large part the roles that students take in the learning process. In the classroom studied here, the didactical contract was not the traditional one in which, on a regular basis, the instructor lectures and the students more or less passively listen. Rather, in this setting, students, working in groups of six for two ninety-minute class periods each week, were given a series of tasks consisting of three problems to be solved, in class, on a weekly basis.

The tasks given to the students varied throughout the semester. The first group of tasks included problems cast in “non-formal” mathematical settings whose solutions involved counting, basic number sense, and geometry. These tasks were experientially based and used little formal mathematical notation. Later tasks moved to a more formalized mathematical setting and included problems from set theory and number theory that are expressed in a more formal mathematical setting using formal mathematical notation. The tasks used in this study included one from the earlier tasks and one from the later tasks.

Within each group, the students would discuss the problems, construct solutions and develop arguments that justified their solutions. After solutions had been constructed and arguments developed by individual groups, the groups presented their ideas to the entire class using overhead transparencies and the whiteboard. Although there was no prohibition for discussion between groups, students rarely discussed their solutions or arguments with other groups prior to the presentations.

The role of the instructor in this class was to set out the tasks and then to respond to the students’ thinking. The instructor accomplished this by joining each group for a short period of time, listening to their ideas and questions and offering guidance and encouragement usually by asking clarifying, real (as opposed to leading) questions about *their* work that probed into the students’ thinking. For example, during presentations, the instructor did not usually ask questions about whether the answers were right or wrong. Rather, he often asked the students how their mathematical argument illustrated the concepts outlined in the problem and what aspects of the argument made it effective. These questions helped students to sharpen their arguments and to highlight particularly interesting or important components of their arguments. Occasionally, the questions drawn out in presentations led to short class discussions led by the instructor and based on ideas or issues that arose in connection with a particular problem.

Researcher's Role

As a researcher, the author participated in videotaping the focus group, but did not contribute to their group discussions. Every class meeting was videotaped, which seemed to make the presence of the camera part of the class surround. This study is a part of a larger study and thus appropriate consents of the participants were obtained as per the guidelines set forth by the university's IRB.

Procedures

Data were collected from two sources. First, data were collected by one camera videotaping the work of the focus group each class period along with any presentations that were made. This produced three hours of videotaping each week. The second source was student-produced artifacts that included student work that was submitted to their instructor for a grade as well as overhead transparencies used in classroom presentations.

Although discussions and presentations were recorded during each classroom session, this study focused on only two tasks discussed by one group. The videotapes and student-produced artifacts were reviewed and analyzed by a faculty researcher and the author. The primary sources of data were videotapes of the focus group working on the tasks. Student-produced artifacts facilitated the comparison of their discussions from the videotape with a written version of their argument prepared to support a public presentation of their ideas and thinking.

Task

The students' work and emerging thoughts analyzed in this study were drawn from the following two tasks. Note that the two tasks are isomorphic, with the second task being given almost 11 weeks after the first.

(1) At a party with five married couples, no person shakes hands with his or her spouse. Of the nine people other than the host, no two shake hands with the same number of people. With how many people does the hostess shake hands? (adapted from D'Angelo & West, 1997, p. 56)

(2) Let $M = \{m_1, m_2, m_3, m_4, m_5\}$ and $F = \{f_1, f_2, f_3, f_4, f_5\}$ with $M \cap F = \varnothing$.
Let $P = M \cup F$.
Let $C \subset M \times F$ given by $C = \{(m_i, f_i) \mid m_i \in M, f_i \in F\}$.
Note: $C \subset P \times P$

Now suppose $H \subset P \times P$ such that:

1. $\forall p \in P,\ (p, p) \notin H$,
2. $H \cap C = \varnothing$,
3. $\forall p_1, p_2 \in P$, if $(p_1, p_2) \in H$, then $(p_2, p_1) \in H$,
4. $\exists!\, h \in P \ni \forall p_1, p_2 \in P$, if $p_1 \neq p_2$, $p_1 \neq h$, and $p_2 \neq h$, then $\left|\{p \in P \mid (p, p_1) \in H\}\right| \neq \left|\{p \in P \mid (p, p_2) \in H\}\right|$.

Find $\left|\{p \in P \mid (k, p) \in H\}\right|$ where $(h, k) \in C$.

(*note* $\exists!\, h \in P$ *should read* $\exists!\, h \in M$)[3]

Strategy of Inquiry

This study used a grounded theory approach to build an analysis of the data. As this analysis began, initial codes were developed from preliminary observations and reflections on the data. These initial codes were refined and extended from further

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[3] The fact that h is in M can be deduced from the question. However, to clarify Condition 4, this note should appear. The students in this study did not have this note but came to the conclusion that h is unique in M by reasoning through the problem.

questions and inquiries that arose. Based on threads seen in the initial codes, an axial coding was designed, which helped make further observations, inquiries, and connections. Following a grounded theory approach, a theoretical perspective emerged that helped answer the guiding questions posed in this study.

Analysis and Coding Procedures

Observations were recorded in field notes as the videotapes were reviewed in order to develop initial codes for the data. These initial codes were based on components of their work that answered the following questions:

1. How did they approach the problem?
2. What aspects of the problem might be intuitive?
3. What conditions were implicit? What conditions were explicit?
4. What was the role of the host/hostess or h/k in their problem solving?
5. How did they view proof? What were they proving?
6. What representations did they build?
7. What conjectures did they make? What questions did they pose?
8. What specific examples did they use?
9. What conclusions did they draw?

After the initial codes were developed for the data, analyzing the student-produced artifacts in conjunction with the videotapes, further connections were drawn between codes in the data. These connections highlighted significant events within each context. These events allowed the data to be analyzed following what Siegler (1996) calls a microgenetic method. This method involves studying the “fine structure of change” (p. 179) that occurs within particular instances of students’ learning and development. Siegler continues that as researchers’ understanding of the changes occurring at any given time become more precise, researchers can better understand the changes made over the short-, medium-, and long-term. He states, “It surely is better to have high quality information about some types of changes than about none” (p. 179). By analyzing the data presented here within significant events, the changes that occur throughout the problem solving process can be better understood and insight can be drawn as to how those changes in their reasoning and justifications affect their proof construction and solution.

Chapter V: Data and Preliminary Analysis

The Handshake Context

Introductory Work and Discussion
(0:00-0:55)[4]

Five of the six students in the group, all but Derek, who had not arrived, had begun to explore the problem on their own for a few minutes prior to the beginning of the video data. During this period, the students were mainly working independently and very little discussion took place. The group's discussion begins with suggestions of how the problem might be approached. On the one hand, Shoshanna offers that they might not need to worry about the host until they figure out everyone else. Then they can "throw him in wherever [they] want." Heather, on the other hand, believes it would be best to start with the piece of information she knows the most about, the information about the host. She proposes using the handshakes of the host as a starting point, then figuring out the handshakes of everyone else. Heather illustrates this by proposing one specific case, "Say the host shook two people's hands and somebody else shook two people's hands." Heather's idea considers how one specific example, involving the host, might influence or force the handshakes of the other party members. The group agrees to follow Heather's idea and decides to look at how the handshakes of the host affect the handshakes of the other party members.

[4] Time codes indicate where each event can be found in the video data.

Emerging Strategies
(0:55-2:25)

Shoshanna combines her idea with Heather’s idea by suggesting that they assign handshakes to people at the party, that they “make this up”. Shoshanna writes two columns, shown in Figure 1.

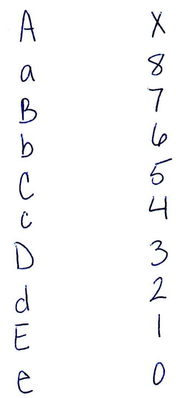

Figure 1. *Shoshanna’s first list*

The first column represents the people at the party where couples are designated by upper and lower cases of the same letter. The second column represents the number of handshakes. As Shoshanna lists the numbers in the second column, the group agrees that the distribution of handshakes must range from eight to zero with each of the nine people other than the host shaking a different number of hands. At this point, it is not clear how, if at all, Shoshanna is imagining a specific relation between the first and second column.

Heather constructs two columns (Figure 2) similar to Shoshanna’s. The only difference is that in Heather’s second column the order of the numbers is reversed.

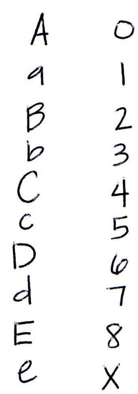

Figure 2. *Heather’s first list*

0:55

Heather indicates that she intends to match each of the people at the party, signified by the capital and lowercase letters A-e, with a number in the second column that represents the number of hands that person shakes. Although Heather’s comments about matching seem to be very explicit, there was no discernible attempt in constructing the lists to display any specific matching. Heather does make a gesture matching the A with the *x*, and designates A, at least tentatively, as the host that shakes *x* hands, where *x* represents the number of handshakes of the host, since this is the number that is still unknown.

Looking at her own list, Shoshanna indicates how she would proceed.

Shoshanna: Well, why don't we try numberin'. Let's say that the hostess shook nine, eight, seven, six, five, four, … let's see if we can do that. See if we can make this up, you know.

Shoshanna: And it might not be possible because maybe, you know, two and three can't be married to each other, you know, type thing.

Heather and the rest of the group show interest in Shoshanna's idea and ask her to continue to explain in more detail.

Shoshanna: So if we can find out that they have to shake each other's hands to make this possible then we can discover that this is not the order that they did it in.

Shoshanna seems to imagine constructing a model, step by step, that would reflect a range of possible handshakes. In this way, she sees the constraints of the problem as helping her determine the possible handshakes. For example, her comment, "And it might not be possible because maybe, you know, two and three can't be married to each other, you know, type thing," considers whether the couple condition, the condition that states that no one shakes hands with his/her spouse, affects the possible handshake assignments.

Shoshanna: [*to Tyler*] So there might be some other way, but that's the only way ... [inaudible]
Tyler: [*to Shoshanna*] [inaudible] have a proof by contradiction
Shoshanna: [*to Tyler*] Yeah, proof by contradiction is all I can think of.

Meanwhile, Heather continues her focus on A and "little a" .

1:47

Heather: Okay, so we know that little a didn't shake hands with the same amount as whatever x is [*pointing to "little a" and then to the x on Figure 2*].

Betsy: [*to Heather*] Well, how do we know that, she could have shook the same as her husband.
Heather: Cause a's are, well, my, no, because no person shakes hands with their spouse.
Betsy: Right, but

Shoshanna now joins the conversation.

Shoshanna:	[t*o Heather*] She could've just done the same amount. [*Shoshanna has now joined Betsy and Heather's conversation.*]
Betsy:	She could've just done the same amount. She could have done x like her spouse.
Heather:	Oh, but only if x is lower than eight.
Betsy:	[*interrupting Heather*] is the total
Heather:	Because if the host shook hands with eight people, she can't. Wait
Shoshanna:	Yeah
Betsy:	Right, she can't.
Shoshanna:	The host can do up to eight people.
Betsy:	Right and she can do up to eight people. And they'll be the two.
Heather:	And still not shake her husband's hand. [*Placing her hand on her forehead, Heather contemplates the truth of this statement.*]
Betsy:	Yeah because she's, yeah
Heather:	Yeah [*with a look of confidence*]

Following Heather's initial statement that the host and hostess cannot shake the same number of hands, Heather, Shoshanna, and Betsy consider the possibility of the host and hostess shaking the same number of hands. Heather believes that the host and hostess might shake the same amount of hands, but only on the condition that "*x* is lower than eight," where *x* represents the number of hands the host shakes.

Shoshanna and Betsy do not initially share Heather's view that the case where the host and hostess both shake eight hands is special in any way. Indeed, they now seek to build an argument that it is possible for the host and hostess to both shake eight hands.

A Key Idea
(2:25-5:34)

For about 10 seconds Shoshanna reflects on the possibility that the host and hostess could each shake eight hands. Then she states:

2:25

Shoshanna: So to do this, wait, eight's not possible either. Oh wait, yes, the hostess cannot shake eight hands FYI because that she would have to shake everyone's hands and she can't shake zero's hands. So eight has to be married to zero.

This last remark transforms the thinking of the group by shifting their focus away from the host and hostess to a more general statement about the couple in which one member shaking eight hands forces the spouse to shake zero hands. The general idea that the spouse of the person that shakes eight hands must shake zero hands will be referred to as the 8-0 idea or the 8-0 relationship. This notation will also be used to describe this specific couple, the 8-0 couple, and the argument that forces that relationship, the 8-0 argument or reasoning. A similar notation will also be used to refer to the other couples at the party, n-k, where n represents the amount of hands that a person shakes and k represents the amount of hands that the spouse of that person shakes.

3:03

Heather agrees with Shoshanna's 8-0 relationship and in the same breath, apparently thinking about a possible pattern, suggests an extension that would organize all couples, i.e. there is a 7-1 couple, a 6-2 couple, a 5-3 couple, and a 4-4 couple. This last couple, she notes, would have to include the host and hence, the hostess. However, Heather's pattern seems a bit premature for the rest of the group who are still focused on Shoshanna's conjecture and her rationale.

Tyler, Betsy, and Heather acknowledge that Shoshanna seems to have a warrant for her claim that the person that shakes eight hands must be married to the person that shakes zero hands. However, it becomes clear that although aspects of Shoshanna's argument seem plausible to the others, they clearly harbor some questions. Betsy is the first to express doubt.

3:20

Betsy:	The one that does eight and zero has to be married? But why?

Shoshanna, in turn, offers a concise explanation.

Shoshanna:	Because eight has to shake everyone else's hands except their spouse. So their spouse has to shake none.

However the rest of the group, echoing Betsy, argues that Shoshanna's result seems much too restrictive.

Tyler:	[inaudible], but the spouse could shake other people's hands.
Heather:	Wait, a minute ago I followed that, but I don't anymore.
Betsy:	[inaudible] But the spouse could, the spouse could shake . . .

Heather responds with her own articulation of the argument for the 8-0 relationship.

3:50

Heather:	It makes sense, [*with a tone of uncertainty*]… No, because you can't shake eight people's h.. , because if they shake that person's hand then that means that that person hasn't shaken no hands. So they can't ever shake, you know what I mean. If you, if they shake eight people's hands, and um, then that means every person that they shook hands with now has one shake.

Heather reframes the argument so that it focuses more specifically on each of the eight people who have shaken hands with the person who shook eight hands.

It is at this point that the focus of the students' work changes from starting with the handshakes of the host and hostess to finding "who the married couples" are. In other words, might a given couple be limited to only a few possible handshake configurations? The group openly acknowledges this shift in focus and addresses what this means for them in terms of counting the handshakes of the host. They know they cannot ignore the handshakes of the host completely but they seem to recognize that they might need to wait before counting those handshakes.

The Solution and the Argument

(5:00-23:04)

For a reason or reasons we do not know, Shoshanna returns to Heather's earlier, seemingly ignored, suggestion about a handshake pattern that would lead to a solution of the problem.

5:00

Shoshanna: You guys, the person who shakes seven has to be married to the person who shakes one because of the fact that z…eight shakes everyone's hands, so one obviously has one. Seven shakes hands with everyone except their spouse and the one that didn't shake hands with anyone. You can keep doing this circular reasoning down until you figure out four is the hostess.

Shoshanna describes her reasoning of the problem as "circular reasoning". What she means is that virtually the same logic that forced the 8-0 relationship between the first couple will also determine a 7-1 relationship between the next couple, a 6-2 relationship between the next, a 5-3 for the next, and finally, a 4-4 for the last couple, thus identifying the host and hostess. Her "circular reasoning" builds from the 8-0 reasoning in that just as the location of the person shaking zero hands was forced from the assumption of his/her spouse shaking eight hands, the location of the individuals that shake one, two, three, and four hands is also forced from the assumption of their spouses shaking seven, six, five, and four hands respectively. Betsy, however, is not satisfied that Shoshanna's assertion is warranted and expresses a need to explicitly work through the "circular reasoning".

5:34

In response, Heather and Shoshanna seek to represent their ideas by listing vertically the people at the party in a column, from E to a, and then, corresponding to each person in the column, listing horizontally all of the people with whom that person shakes hands. (See Figure 3.) Heather and Shoshanna begin by listing the handshakes for E and e.

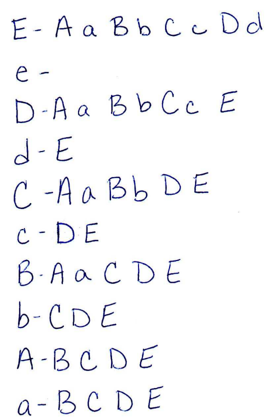

Figure 3. *Heather and Shoshanna's second list*

While Shoshanna and Heather construct their list (Figure 3), they reflect on the purposes their representations serve. Shoshanna explains that the representation shows rather than proves. For Heather, the representation is an outline of whose hands everyone at the party shakes; it captures the global situation.

8:23

Even as Shoshanna and Heather complete the representation that anchors their arguments about the distribution of handshakes, the opportunity arises to test the potential of their representation for clarifying the 8-0 argument. In this way, Shoshanna responds to Tyler's lingering doubt about the necessity of the 8-0 relationship.

8:23

Tyler:	So why do you know that e didn't shake anyone's hands?
Shoshanna:	Um, because, so we know that, because A has shaken, E has to shake everyone's hand, right [*pointing to Figure 3*]? Well, E has to be married to zero because now everyone has shaken hands. You see, these all have E [*pointing to the list of letters E to a in the first column of Figure 3*]. That's why it makes sense. So, here's the group right, so …
Tyler:	But like someone down here could …
Shoshanna:	[*pointing to Figure 1*] So E shakes hands with him, E, E, E, E, E, E, E [*writing the letter E next to each letter, except for E and e, in the first column of Figure 1*].

Shoshanna:	So here's the couples. Here's the eight couples, or the eight people, right, other than him and his wife. If E shakes hands with all eight of these people, right, then that's the only way he could have eight handshakes right? Then…

Shoshanna:	So, we have these eight people right? Well everyone's now shaken one hand, right? Who's the only person who cannot shake hands? The person who didn't shake hands with him.
Tyler:	Oh, him, and that would have to be the spouse.

Shoshanna explains the 8-0 argument by first pointing out the occurrence of E in each row of Figure 3, except the rows corresponding to E and e. Thus, every person, except e, has clearly shaken at least one hand. She further emphasizes this point by modifying her first list, Figure 1, to show the results of E's handshakes. She places the letter E next to each person in Figure 1 with whom E shook hands. (See Figure 4.)

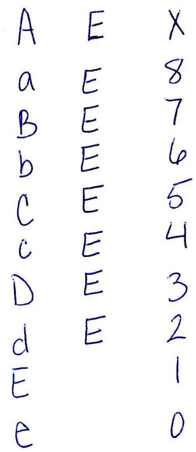

Figure 4. *Shoshanna's modification of Figure 1*

9:38

Shoshanna now turns to the argument for the couple in which one member shakes seven hands. Shoshanna returns to her second representation, Figure 3 to anchor her argument. She explains that someone other than E or e must shake seven hands, and since D is next in line in her representation, she assigns seven handshakes to D. She then points to the letter D and the letters in the row corresponding to D representing the handshakes of D. Shoshanna relies on Tyler's interpretation of her representations to provide the warrant that she cannot seem to fully express verbally.

Betsy raises the question of uniqueness by asking "but could we do it differently?" Shoshanna replies, "No, there's no way else that someone could shake eight hands unless they were married to someone that shook no hands."

11:30

Reflecting on her experience so far in seeking to provide a compelling way to present her reasoning, Shoshanna prepares a new representation to display “two different ways of looking at who each person shakes hands with”. (See Figure 5.)

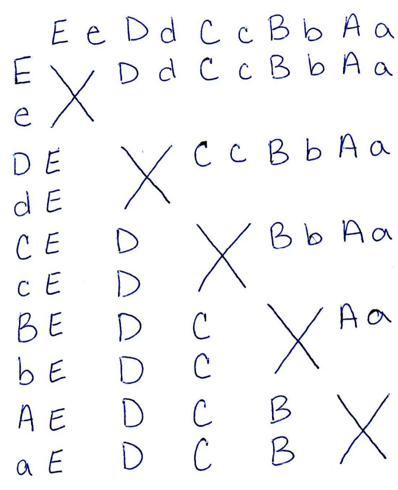

Figure 5. *Shoshanna’s final representation*

Each row of this representation, like Figure 3, shows with whom a given person shakes hands, thus establishing the relationships that were unknown in Figure 1. The columns, however, provide a sense of each particular person’s handshake situation at any stage of the argument. In this way, Figure 5, seen as an extension of Figure 4, effectively anchors the layered process of their complete reasoning. The large X’s in the figure cross out handshakes that are not permitted. Shoshanna says this representation makes it easier to see why her solution works.

Heather and Shoshanna both reference the roles of the representations for them as they build their presentation. Referring to Figure 5, Shoshanna notes:

> Shoshanna: The way this makes more sense to me is it shows that everyone's shaken hands, so that it has to be married to that person. You see, because E has to do eight people's hands, it shows that the only person who hasn't shaken hands with, the only person who hasn't shaken hands period, is his wife.

Heathers, referring to Figure 3, adds,

> Heather: I keep having to look at this to reprove it to myself that that's true.

Indeed, these students found the representations they constructed to be indispensable to not only their own understanding but also to how they could build a compelling argument.

> Heather: I think the best way, yeah the best way to represent it is not with words, it's just with the charts and then we can explain it when we're up there.

The two representations, Figure 3 and Figure 5, become the focus of the group's class presentation as they invite the audience to reason, along with them, from the representations.

The Set-Theoretic Context (11 weeks later)

Introductory Work and Discussion
(Tape 1, 7:15-16:13)

Tape 1, 7:15
For the first 9 minutes, the students work through the meaning of the notation – unions, intersections, and cross products. They focus mainly on the general meaning of cross products, as well how they apply to the context of the problem. In the students' explanations of cross products, several of them give examples of ordered pairs that are found in the cross product of MxF. The students refrain from analyzing the four conditions of the problem until they have built meaning for the descriptions of the sets. Once they are comfortable with the notation of unions, intersections, and cross products, they apply their understanding to the first three conditions. The first condition states:

1. $\forall p \in P, \ (p, p) \notin H,$ [*Condition 1*]

From this condition, the students conclude that you cannot have an ordered pair in the set H where both elements in the ordered pair are the same. Next, they consider the second condition:

2. $H \cap C = \varnothing,$ [*Condition 2*]

This statement leads them to the conclusion that an ordered pair of the form (m_i, f_i) cannot be in H, where the index of both m and f are the same. The third condition states:

3. $\forall p_1, p_2 \in P, \ \text{if } (p_1, p_2) \in H, \text{ then } (p_2, p_1) \in H,$ [*Condition 3*]

The students conclude from Condition 3 that since no elements of the set C are in H, where $C = \{(m_i, f_i) \mid m_i \in M, f_i \in F\}$, then any ordered pairs from the set $\{(f_i, m_i) \mid f_i \in F, m_i \in M\}$ are also eliminated from the set H. In other words, if the elements m_i and f_i

contained in the ordered pairs of C are reversed, then those new ordered pairs cannot be in H.

Summarizing the three conditions in their own words, the group concludes: You cannot have an ordered pair in H where both elements have the same index.

Emerging Strategies

(Tape 1, 16:13-Tape 2, 4:08)

The students focus their work on the fourth condition, seeking to make sense of how Condition 4 relates to the set H and the first three conditions. The fourth condition states:

4. $\exists!\, h \in P \ni \forall p_1, p_2 \in P$, if $p_1 \neq p_2$, $p_1 \neq h$, and $p_2 \neq h$, then $\left|\{p \in P \mid (p, p_1) \in H\}\right| \neq \left|\{p \in P \mid (p, p_2) \in H\}\right|$. [*Condition 4*]

For 30 minutes, the students work to build meaning for Condition 4. They recognize that central to that meaning will be understanding the cardinalities presented in Condition 4. Thus, they focus on how they might describe, in their own terms, those cardinalities. Heather and Betsy conclude that cardinality in this problem means "how many ordered pairs [we're] gonna have". Now, to find the cardinality of a set of the form $\{p \in P \mid (p, p_1) \in H\}$ for a given $p_1 \in P$, requires determining all the ordered pairs in H that have p_1 as the second entry. The students identify each of those ordered pairs with the first entry of the ordered pair. They refer to cardinality in this setting as "the number of p's that go with p_1".

Using this language, Heather asks, "Why could you have fewer or more p's with one of the elements than you could with the other?" In response, the students consider fixing p_1, "what if p_1 is m_1?" From all possible ordered pairs ending in m_1, they make use of the first three conditions to eliminate two ordered pairs as possible members of H. In particular, “[the] p [that goes with m_1] cannot be f_1 or m_1”. “Now that limits p to only eight options [namely m_2, m_3, m_4, m_5, f_2, f_3, f_4, f_5].”

Heather generalizes the argument above for every choice of p_1 and asks, “All we want to do is count them, so can’t we just say it’s eight and we’re done?” Heather proposes that irregardless of the particular choice for p_1, the cardinality of the ordered pairs in H that end in p_1 would be eight. This proposal elicits a brief discussion of the purpose of examining cardinalities. Heather states that her goal is to figure out why you could put one element with p_1 but not with p_2. In other words, why the two cardinalities corresponding to p_1 and p_2 are not equal.

46:52

To anchor more globally the kind of thinking that has emerged, Derek suggests that a list might be helpful. Everyone agrees that a representation could afford them a view of the structure these conditions demand. Derek argues that the list would make it easier to go through the elements of PxP and see if the conditions are satisfied.

Several suggestions are put forward for an appropriate list: PxP, H, and the complement of H. Betsy pursues PxP in spite of the objection of some that it is too large. Shoshanna starts trying to build a representation for the elements in H and the elements not in H simultaneously. Tyler, Derek, and Heather follow a different idea: create a smaller problem situation that resembles the original. In this new problem, M and F each consists of two elements: $M = \{m_1, m_2\}$ and $F = \{f_1, f_2\}$. Tyler states that by limiting these sets it would be easier to work with the ordered pairs.

Heather and Derek begin to list the elements of PxP in their problem.

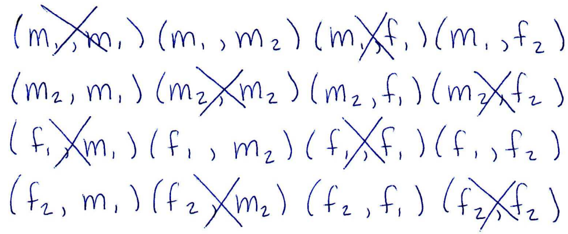

Figure 6. *Tyler, Derek, and Heather's small version of PxP*

They cross out ordered pairs that violate the first three conditions of H, leaving two ordered pairs in each column. (Note that each column lists the ordered pairs that have the same second entry.) At this point, they conclude that the cardinality of each subset (column) in their smaller problem is at most two.

Heather applies Condition 4 to their results.

57:44

Heather: Ummm, so this is saying that you can choose one of these f_1, f_2, m_1, m_2 and there will be less, there will be fewer ordered pairs for that one than for another one? That isn't true.

Heather alerts the others that Condition 4 is not satisfied. Derek responds by eliminating the crossed out ordered pairs from Figure 6 and crossing out ordered pairs in the new array so that there are only two subsets (columns) that have the same cardinality.

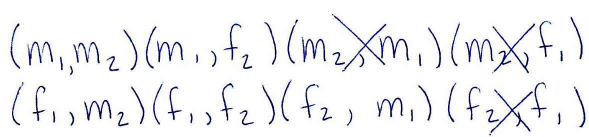

Figure 7. *Derek's elimination according to Condition 4*

Thus, the subsets (or columns) have the following cardinalities: 2, 2, 1, and 0. Note that this is the first instance where the possibility of a zero cardinality arises.

A Key Representation

(Tape 2, 4:08-19:44)

While Derek, Tyler, and Heather are working with a smaller "population," Betsy takes on the task of listing out all 100 elements of PxP. Her list is shown below as Figure 8. The elements crossed off are the ordered pairs that have the same index, a result of applying Conditions 1-3. Note that Betsy's array is organized so that the second elements in the ordered pairs are the same within a row.

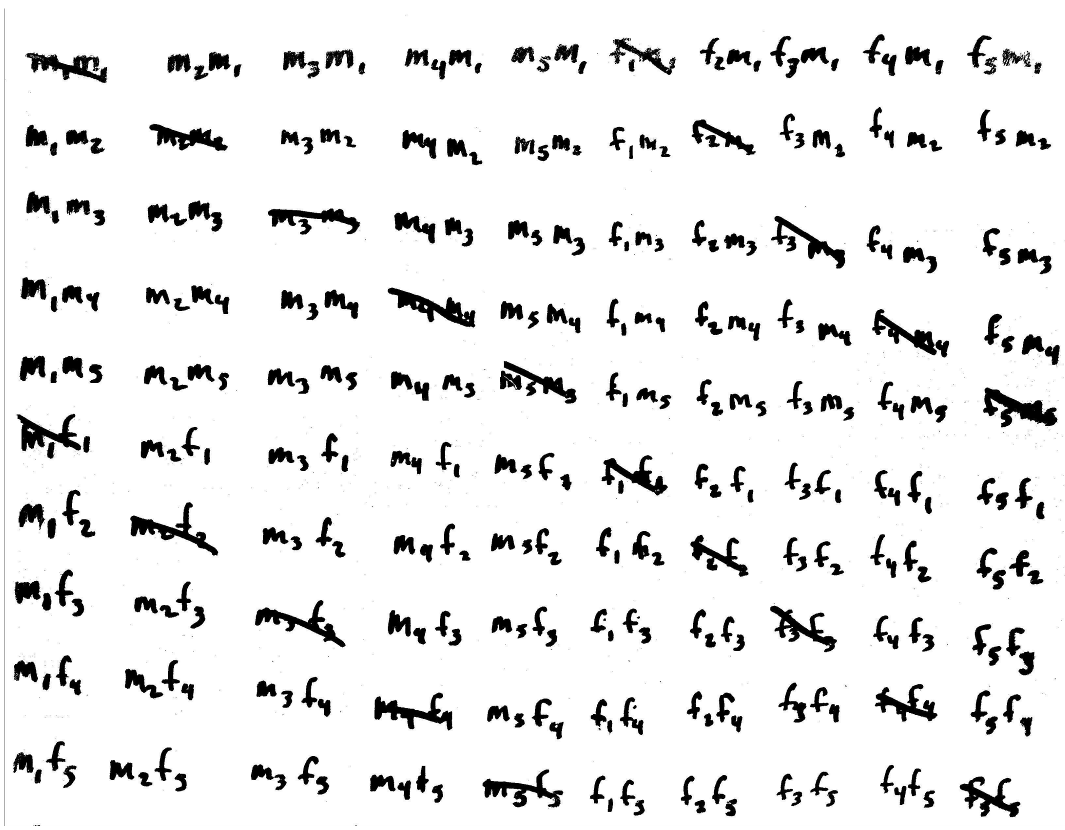

Figure 8. *Betsy's Array of PxP*

With Betsy's array on the table, the whole group returns to the original problem. Shoshanna draws the following conclusion from Derek's work and the new array before them:

4:52

Shoshanna: So pretty much, we have to cross them out so that no possible p_1's and p_2's can be equal. We have to cross out somehow that way

Note that Shoshanna's statement, "no possible p_1's and p_2's can be equal," means that the number of ordered pairs ending in a given p_1 and p_2 cannot be equal.

12:54

Row by row, Shoshanna crosses out ordered pairs so that the number of ordered pairs remaining in each row, reading down the rows, is: 8, 8, 7, 6, 5, 4, 3, 2, 1, 0.

Although Shoshanna's structure correctly reflects the fact that only one cardinality can be duplicated, it leaves the same number of ordered pairs in the first two rows, which, the group realizes, means that h is not uniquely determined. Indeed, h could be either m_1 or m_2. Since h must be unique in M, they must consider a second case. Shoshanna rebuilds the array so that the first and the fifth rows (the rows with ordered pairs ending in m_1 and f_1 respectively) are the same and h is unambiguously m_1.

17:05

At this point, although the group seems pleased with their progress, there still might be questions. Indeed, Shoshanna asks about the cardinality associated with h. Does h have a cardinality of eight or could it be seven or some other number. After reflecting on Shoshanna's representation, Derek notices that Condition 3 is not fully satisfied. In particular, he notes that if one ordered pair, say (m_i, f_j), is eliminated, then they have to eliminate the "flip" of that ordered pair, namely (f_j, m_i). Tyler concludes that they need a "more structured elimination".

The Solution, Argument, and Connection

(4/11/05 Tape 2, 19:44 – 4/13/05 Tape 1, 7:25; 4/13/05 Tape 2, 5:10 – 15:57)

For the third time, Shoshanna begins to structure her array. However, this time, in response to Derek's observation, she chooses to indicate specifically what ordered pairs are kept as well as those that are to be eliminated. She circles the entire first row, including the two ordered pairs that had been crossed out, marking the ordered pairs that are to be retained. Then, turning to the first column and following Condition 3, she circles each of the uncrossed-out ordered pairs individually. As before, the cardinality associated with m_1 is eight. While others propose moving on to cardinality seven or other ways of eliminating ordered pairs, Shoshanna examines the first column. Indicating by a sweeping gesture the row associated with f_1, she makes a remark.

Shoshanna: This has to be the one that's empty. It is the only one that has it crossed in this row [column].

Thus, she crosses out each entry of the row and column that contain f_1. The class period ends and Shoshanna takes a copy of the array with her, confident that her elimination pattern will be successful in determining H.

4/13/05 Tape 1, 0:00

The next class meeting, Shoshanna proposes a solution on her result from the previous class, i.e. the m_1 row has eight ordered pairs in H and the f_1 row has no ordered pairs in H,

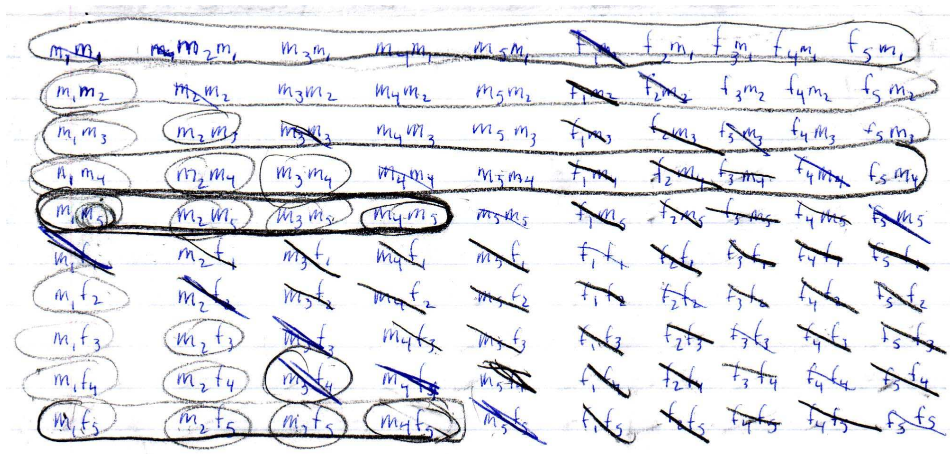

Figure 9. *Shoshanna's marked array*

Shoshanna explains that regardless of which row they choose for a given cardinality, "eight has to be with zero, seven has to be with one, six-two, five-three, four and four, same as the handshake," establishing that the rows with the same cardinality, four and four, point to the elements h and k. Note that Shoshanna's pairing of rows, 8-0, 7-1, 6-2, 5-3, and 4-4, is according to the subscripts of the elements, m_1-f_1, m_2-f_2, m_3-f_3, m_4-f_4, and m_5-f_5. Although her proposal refers to the surface counting features of her solution and the underlying argument drawn from her work the previous day, there is, with the exception of Betsy and Heather, little or no evidence of serious concentration on the arguments. Instead, Derek, Tyler, Shoshanna, and to an extent Betsy, become involved with a detailed, careful examination of the extent to which this problem is the "same as the handshake [problem]".

4/13/05 Tape 1, 1:32

Derek's, Tyler's, and Shoshanna's careful examination begins with a comparison of each element and set to the problem in the handshake context. For example, the set M corresponds to the males at the party, F to the set of females at the party, and C to the set of couples at the party. They then check that the four conditions of H can be mapped to the conditions of the problem in the handshake context. They establish that the first condition corresponds to the idea that you cannot shake your own hand. The second condition corresponds to the condition that you cannot shake your spouse's hand. They describe the third condition by stating, "If I shake your hand, then you shake mine." They match the final condition, Condition 4, with the statement, "Of the nine people other than the host, no two shake hands with the same number of people." Comparisons between the two contexts continue until an isomorphism is established. Surprisingly, the students do not want to use this isomorphism in their public presentation. Rather, they choose to develop an argument for their solution anchored in their representation. Following the argument, they will bring in the isomorphism as a postscript.

4/13/05 Tape 2, 5:10

The group begins their presentation of this problem by first familiarizing the class with their array, explaining how the ordered pairs are arranged in the array and how the array represents PxP. They then go through the first three conditions and eliminate ordered pairs as they did during their group work. Derek and Shoshanna then begin to explain Condition 4.

Derek:	And then number four, number four basically tells us that each um, element, the number of elements in H that end in a given, the number of pairs in H that end in a given element has to be unique except for the little h, so… [*Derek pauses.*]
Shoshanna:	So, so what it means in English is the second pair right here [*pointing to the element m_1 contained in the ordered pair (m_1, m_1)*], there can only be so many, and that's this row right [*pointing to the first row in the array*]? You understand that? There can only be, there has to be a unique number in each of these rows except for the row h. Where h equals let say m_3. This row can be the same as one other row. We'll show you how we're gonna do this.

Derek tries to explain Condition 4 as they had in their group work but hesitates in finishing his explanation. Shoshanna jumps in, feeling the need to "translate" Derek's explanation in a way that will help the students better understand Condition 4. To do this, she relates the rows of the array to "the number of pairs in H that end in a given element." She then focuses her discussion on rows rather than ordered pairs. The elimination according to Condition 4 then is explained in terms of the rows and columns of the array as discussed in their group work above. Although the group did not want to explain the problem in terms of the handshake context, they still found a need to rely on the context of their representation in order to help their classmates understand the problem, the solution, and the argument.

Chapter VI: Analysis

The analysis presented here will build from the students' use of representations as discussed in the previous chapter, allowing for an examination of the students' reasoning in each context.

Handshake Context

The analysis in the handshake context will connect with four representations displayed in the last section but recapitulated together in Figure 10 for ease of reference.

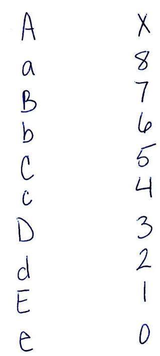

Figure 1. *Shoshanna's first list*

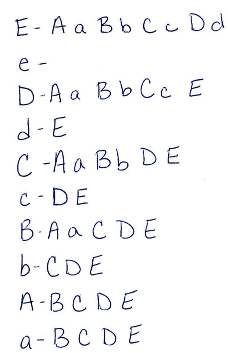

Figure 3. *Heather and Shoshanna's second list*

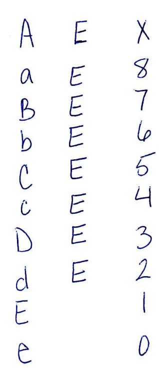

Figure 4. *Shoshanna's modification of Figure 1*

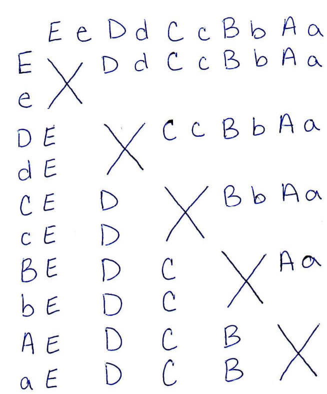

Figure 5. *Shoshanna's final representation*

Figure 10. *Student representations in the handshake context*

From the very beginning, the students were confident in their understanding of the statements and conditions in the problem, i.e. what constitutes a handshake, the symmetry of a handshake, and precisely what the question asks. Thus, as they sought to solve the problem, their focus was drawn first to the relationships between those statements and conditions. However, no notation was suggested for the students to use. Hence, in order to reason effectively, they needed to invent useful notation and perhaps even build representations that would communicate, anchor, and advance their reasoning about the problem. Thus, the context of the problem influenced the choices they made in their initial work. Given the richness of their understanding of the problem, the initial reasoning of these students was guided by a strong sense that the distribution of handshakes might take on many different possibilities. Indeed, it was not clear to them that there even needed to be a unique solution to the problem. The first two written representations, produced by Heather and Shoshanna (Figures 1 and 2), presented the possibility of identifying people with numbers of handshakes, but without indicating any particular identification. Heather further suggested that by first considering the handshakes of the host the handshakes of the other party members might be constrained and thus the complexity of the problem reduced.

Shoshanna's contradiction strategy of assigning handshake values and simultaneously looking for contradiction of one of the conditions was designed with the purpose of furthering this reduction by suggesting that from specific examples she might infer information about a potential distribution for handshakes. Her strategy suggested more than just eliminating possibilities. She utilized a specific case to clarify how the constraints of the problem might reveal ways that the problem could be (or even needed to be) structured. Indeed, Shoshanna's response to realizing that the host and hostess could not both shake eight hands was not to open another possible case, but rather to continue to structure handshakes according to the relationship she found.

Heather communicated her understanding of Shoshanna's contradiction strategy as she claimed that the host and hostess could not shake the same amount of hands since "no person shakes hands with his or her spouse". Heather believed that if the host and hostess shook the same amount of hands, they would be forced to shake each other's hands. Under questioning, Heather refined her case, stating that the host and hostess might shake the same amount of hands, but only if "x is lower than eight," where x represented the number of hands the host shook. Indeed, she believed that such a case would force the host and hostess to shake hands.

Shoshanna realized that she had to agree with Heather's point, i.e. the host and hostess cannot both shake eight hands. She also realized that her argument for Heather's point had been built around a key idea and, in fact, resulted in a much stronger assertion. She stated that not only can the host and hostess not each shake eight hands, but if *any person* shakes eight hands then the spouse of *that person* must shake zero hands. By virtue of this argument, Shoshanna had shifted their purposes, from this point on, away from finding the handshakes of the special host/hostess couple as a starting point to finding the relation of the numbers of handshakes within arbitrary couples.

Shoshanna offered a partial warrant for the 8-0 assertion: the person that shakes eight hands must shake everyone at the party's hand except for him/herself and his/her spouse, therefore the spouse must shake zero hands. She did not, however, clearly warrant the last inference. Therefore, her initial warrant did not sufficiently clarify the logical necessity of the 8-0 idea. The other students, clearly uncertain of the validity of her warrant and seeing her result as perhaps infringing too much on their expectations of a wide range of possibilities for handshakes, challenged her statement and the warrant she put forward to back it. Could it be true that the handshakes had to be so strictly structured?

Very quickly Heather commented that Shoshanna's idea/argument made sense. Heather was still working through an argument for the 8-0 necessity in her own mind. Heather's understanding became apparent in her own characterization of the logical necessity of the 8-0 idea: when *one person* shakes hands with eight people, each of the eight people that *that person* shakes hands with now has one handshake, leaving only one person in a position to shake zero hands, the spouse of *the person* that shook eight hands. Heather's explanation subtly shifted the focus from the person shaking eight hands to each of the eight people whose hands were shaken, i.e., the eight people that now had one handshake. With this shift, it becomes clear that there is only one possible person who could shake no hands. Shoshanna indicated her acknowledgement of the need to clarify the warrant for her assertion, stating, "I knew it had a reason, but that's why."

Once the argument for the 8-0 couple had been settled, at least for Shoshanna and Heather, they moved on to the remaining couples. Here Shoshanna and Heather recognized that representations would be necessary in order for them to construct clear warrants for their arguments. For them, the representations did not prove their assertions; rather they offered a backdrop from which their reasoning could be formulated, communicated, and examined. Indeed, Heather and Shoshanna built their representations, particularly Figure 3 and Figure 5, hand in hand with their arguments about the handshake relationships. Thus, their representations became an anchor for the argument they were developing.

At one point, the uniqueness of their solution was questioned. Shoshanna returned to the necessity of an 8-0 couple and its warrant to argue that the structure for their solution had to be fixed. Indeed, Shoshanna's representations evolved as she revised her existing representations to clarify the reasoning for the 8-0 idea. As she explained to Tyler why E shaking eight hands necessitated the spouse of E shaking zero hands, Shoshanna related Figure 3 to her earlier list, Figure 1, and modified it, as shown in Figure 4, so that Tyler would be able to see why the spouse was the only one that was left to shake zero hands. Indeed, Betsy stated in reference to Shoshanna's array, "Her chart, that's where it clicks." A representation was needed to establish the warrant for the group.

Shoshanna's new array, Figure 5, reprised Heather's way of warranting the 8-0 relationship. Although both Figure 3 and Figure 5 are essentially the same in that each row contains the same letters, the arrangement of those letters communicates a view of the two perspectives of the 8-0 argument. For instance, on the one hand, the rows of Shoshanna's array show the eight specific people with whom person E shook hands. On the other hand, the columns of her array show that each of those eight people has shaken hands with person E.

Heather's statement, in regards to Figure 3, "I keep having to look at this to reprove it to myself that that's true," indicates the difficulty of overcoming her intuition that there were many possible handshakes distributions. Indeed, this intuition is completely contradicted by the logical necessities of the problem's conditions. This intuitive suspicion persisted for some (particularly Tyler, Heather, and perhaps Betsy) throughout their work. Thus, they continually relied on their representations in order to override their intuition. Indeed, their representations anchored the reasoning of their solution.

Set-Theoretic Context

The first striking feature of the problem posed in the set-theoretic context was the use of formal, some might say, abstract, notation. Thus, the students' initial purpose was to construct working meanings for the notation, definitions, and conditions used to pose the problem. For the first 40 minutes of student work on the problem, specific examples were often used with the purpose of building meaning but no formal lists, arrays, or representations that might be construed as connected with the problem were introduced.

Although the students constructed solid meanings for the first three conditions, the full effects of Condition 3 were not immediately evident. They quickly noted that this condition, in connection with Condition 2, states that if a particular ordered pair from C, such as (m_1, f_1), is not in H, then the ordered pair with the elements reversed, (f_1, m_1), is not in H. But they were not yet prepared to make use of the fact that it states that if *any* ordered pair, such as (f_3, m_5) is in H, then the ordered pair with the elements reversed, (m_5, f_3), is also in H. Thus, the fact that Condition 3 would need to be applied much more widely than to the set C was to come later as an important insight.

Examining Condition 4, the students understood the meaning of cardinality, but did not yet recognize how to meet the requirements of the condition in a way that would help them solve the problem. Indeed, from their initial conclusion that any of the cardinalities expressed in Condition 4 could be at most eight, they actually contradicted

Condition 4 by supposing that those cardinalities might be equivalent. The students' main concerns up to this point (about 38:00) were:

1. What does it mean for the element h to be unique?
2. Why can't the cardinality associated with two elements be equal?
3. What are we trying to find?

The first two questions focus on applying the meaning of the notation to the context of the problem. The last question seems to center on a purpose for their work. It suggests that perhaps these students were seeking to understand the structures of the problem without thought to imagining how they might seek to answer the question the problem posed.

The students built three lists in which they were able to present the first three conditions by either listing or eliminating those ordered pairs with a common index. However, their limited grasp of the meaning of Condition 4 hindered their ability to illustrate the fourth condition in a representation. Thus, Derek constructed and manipulated a mini-case representation with the purpose of understanding more explicitly the fourth condition. Although Derek did not accurately apply Condition 3 to his mini-case representation, he did obtain significant insight into how Condition 4 might structure the set H. He then applied this understanding to how he might use Condition 4 to structure H in the original problem.

Betsy's global representation (Figure 8) offered a valuable perspective on Condition 4. The structure of the 10x10 array provided a view in which the rows of the array were seen as presenting separate subsets of H, each of which represented all of the ordered pairs that end in a particular member of P. The rows also conveyed to the students that since nine rows of the array must present different cardinalities, up to a maximum of eight, one of the rows must have zero ordered pairs from that row in H. Recall that the possibility of a zero cardinality first arose in Derek's list where the columns of his representation offered a similar perspective to the rows of Betsy's array, illustrating the necessity of a zero cardinality. This idea, that was key in determining which of the elements of the rows were to be included in H, also turned out to be key in their solution to the problem.

The group used several different strategies with Betsy's representation in order to understand the set H. Anchored in Derek's elimination from his mini- representation, Shoshanna first eliminated ordered pairs from H so that the two cardinalities that were the same were the highest possible value, i.e. the cardinalities associated with m_1 and m_2 were both eight. Although, due to the uniqueness of the element h, these cardinalities were later adjusted so that the cardinalities associated with m_1 and f_1 were the same; those cardinalities were still both eight. This choice of a duplicated cardinality is interesting, particularly where m_1 and f_1 are the same, in that the same example was considered in the earlier handshake problem context.

Questioning the uniqueness of the solution forced the group to focus on the key ideas of the problem. Indeed, this very question led Derek to realize that Condition 3, the symmetry condition, had not been fully applied. This recognition resulted in a new method of elimination that pointed directly to the idea that if the cardinality associated with an arbitrary m_i is eight then the cardinality associated with f_i must be zero (similar to the 8-0 relationship that emerged as key in the handshake problem).

Although the group first constructed representations for the purpose of better understanding the set H, they realized as soon as they had been successful in their construction that their final representation actually gave them the solution to the problem–they could say that the cardinality associated with k was 4. Although the reasoning for that solution could also be drawn from their completed representation of H (as was seen later in the student' public presentation), the students initially chose to justify the solution for themselves by developing, and then appealing to, a carefully detailed isomorphism between this problem and the earlier handshake problem. The students worked very carefully in a systematic way to identify their mapping and then check that it preserved the key conditions that defined the two problems. Having done this, they reasoned that the solution to the earlier problem as well as the reasoning behind it could then map directly to this new setting. Nonetheless, to provide their classmates a sense of the great effort put forth to solve the problem, the students chose to publicly explain the argument and solution strictly within the set-theoretic context, anchoring their argument in their representation and only then to share the isomorphism.

Chapter VII: Discussion and Conclusions

Discussion

The analysis offers the following insight in responding to the guiding questions of this study.

1. How do the students' purposes and choices affect their use of particular representations in each context?

In the handshake context, the students' purposes centered on determining the number of handshakes of the hostess by using both explicit and implicit relationships associated with the problem to imagine a distribution of all handshakes. Although they wanted to find the handshakes of the hostess, they could not attack that problem directly. They first needed to build a structure that verified the handshakes of the hostess. For example, their purpose became using the number of handshakes of the host to find the number of handshakes of the hostess. Shoshanna's suggestion to assign handshakes to everyone at the party, using Figure 1, indicated that the handshakes of the host and hostess could then be used to find the handshakes of all of the party members. Shoshanna's contradiction strategy took a deeper look at the relationships, considering how the conditions would affect those handshake assignments, i.e. the condition that no one can shake the hand of their spouse. Indeed, the handshake context offered the students a wide span of relationships to work with as they searched for the solution.

The students' strong intuitive understanding of the relationships, concepts, and ideas in the problem facilitated their ability to generate and use examples. Although many examples were available for consideration, Shoshanna posed a contradiction strategy that limited the possible cases that the group could consider. Heather's focus then narrowed as she searched for contradictory examples that might eliminate impossible cases, i.e. the host and hostess each shake eight hands. Shoshanna used the relationships found in the problem to not only eliminate the 8-8 case, but to structure the solution from the 8-0 idea.

The set theoretic context was described by notation that the students did not understand beyond mere definitions. The students did not have a sense for how those definitions might be used in a meaningful way or what questions would be helpful in answering. Thus, their initial purpose was to understand the problem.

The examples that the students used in the set-theoretic context were first generated to illustrate the meaning of the first three conditions. Their examples did not immediately give them further insight into how the conditions structured the problem, but did allow them to see how a given condition played out in a visual way, i.e. the ordered pair (m_1, f_1) cannot be in H.

The group's approaches in both contexts follow Moore's (1994) *Concept-Understanding Scheme*. His model of students' concept understanding illustrates that some of the major sources of students' difficulties in doing proofs arise from their inability to understand and use the language and notation and their inability to generate and use examples. By viewing proof as a problem solving activity, Moore's model can be adapted to this study and offer a helpful way of discussing the understanding of these students. In the handshake context, when the students understood and were able to use the language and notation, they were focused on the solution and strategies for finding the solution. Their purposes, as previously described, as well as the many relationships available to them allowed them to generate and use examples that were directed towards finding and justifying the solution. In contrast, in the set-theoretic context, these examples and strategies either were not present or were much delayed. The students were limited in their choices as a result of their difficulty in understanding the relationships in this context.

The relationships found in the handshake context led the students to focus on the 8-0 idea as well as the warrant for the 8-0 argument in order to find the handshakes of the host and hostess. Their arguments and representations were then made in order to clarify the concept and the importance of the 8-0 idea. Indeed, Heather and Shoshanna's representation in Figure 3 was built to illustrate how the 8-0 idea extended to the other couples, particularly the host and hostess. Recognizing the importance of the warrant for the 8-0 idea, Shoshanna built an array shown in Figure 5 that clarified that warrant, extending it to the other couples, particularly the host and hostess.

In the set-theoretic context, the group did not rely on any representations that were written for nearly 40 minutes. As previously stated, the students' purpose was to understand the problem. They finally chose to use representations in order to fulfill that purpose and understand what the set H looked like. Derek stated that a representation would allow them to go through and check to see if each of the four conditions were satisfied in the set H. The students' in the group produced three written representations with that same purpose, to see what H looked like.

2. What do those representations tell us about their reasoning in each context?

The representations in each context show us how the students moved between local and global perspectives in order to solve the problem. The question in the handshake context, "With how many people does the hostess shake hands?" is a local question focused on the handshakes of one specific person at the party. However, the students attacked that local question by first looking at the global picture, focusing on how the relationship between the constraints and the couples might determine all of the handshake assignments. The group had then approached that global picture by taking a localized look, namely the 8-0 idea. In the set-theoretic context, the students focused on understanding the problem and set out to do so by looking at the global picture. They finally decided to use representations in order to see that global picture, i.e. to better understand what the set H looked like. Thus, the group was focused on the global aspect of the problem, with little thought to any localized aspects of the problem.

In building their representation shown in Figure 3, Heather and Shoshanna first sought to demonstrate the reasoning behind the 8-0 idea, and then use that idea to illustrate the reasoning for the other handshakes. Shoshanna's representation, Figure 5, was also built with the purpose of clarifying the warrant for the 8-0 idea. It is clear from the focus of these representations that the students placed the 8-0 idea at the foundation of their reasoning for the solution. Indeed the 8-0 idea offered both conviction of the structure and solution of the problem as well as understanding of the justification for the solution.

Raman (2003) referred to such an idea as a *key idea*, explaining that key ideas allow a student's heuristic (informal) ideas to be seen in a rigorous, formal proof. The use of key ideas is thus essential if students are to develop a mature view of mathematical proof. When Shoshanna discovered the 8-0 idea, she was able to recognize it as a key idea and build her reasoning for the solution from that idea.

The transformation of the students' representations throughout their work demonstrates their ability to change their strategies and their focus as their reasoning developed. This proved to be instrumental in using the 8-0 key idea to structure their reasoning. Indeed at the discovery of the 8-0 idea, the group's focused turned from the handshakes of the host and hostess to the handshakes of any arbitrary couple. This change in perspective was necessary for them to solve the problem. Shoshanna's final representation, Figure 5, focused on the warrant for the 8-0 idea which required an additional shift in perspective, one that focused on the perspective of the people whose hands were shaken.

Raman (2003) discussed key ideas in terms of their use in mapping heuristic ideas to rigorous, formal proofs. Now as these students prepared their presentation of the solution, Derek asked how the reasoning would work if there were n couples. Shoshanna stated the solution, that the person who shakes the most hands would be married to the person that shakes no hands and the rest of the handshakes follow the same pattern as in the case where $n = 5$. Derek then concluded that they would conduct a proof by induction. Thus, the 8-0 key idea allowed them access to a mapping of their ideas to a more general problem and a more formal proof.

The representations in the set-theoretic context were focused on clarifying how the conditions affect the set H. Thus, the use of a key idea to structure their reasoning only came into play as they came to understand the set H and as they considered possible solutions. Although the representations in this context were not focused on a key idea, they illuminated the reasoning that the students considered as they worked to understand the set H. Many group members dismissed Shoshanna's lists of what might be in H and what is not in H in favor of Derek and Tyler's representation and Betsy's array, each of which offered further insight into how Condition 4 affected the set H. Indeed, Derek and

Tyler's smaller representation signaled to the group what Condition 4 meant and, most importantly, how they could apply it the larger set in Betsy's array. However, their inaccurate application of Condition 3, the symmetry condition, hindered, to some degree, the group's understanding of the structure of H. Fortunately, the symmetric structure of Betsy's array signaled to the group the need to more accurately apply Condition 3 to their elimination strategy.

The group's search for a representation that would indicate how the conditions might structure the set H seems to indicate that they were trying to figure out what the conditions were telling them to do, as if the conditions were steps to solving a problem. In fact, the group used the work from Derek and Tyler's representation as a set of instructions to be applied directly to Betsy's array. Only when these instructions failed to produce a correct solution did the group turn to alternate methods of elimination drawn from the relationships present in the problem and illustrated in Betsy's symmetric array.

3. How do their representations and reasoning compare between the two contexts?

The students in this study were given two mathematically isomorphic problems. Their work resulted in two isomorphic[5] representations. Although their final representations in both contexts were isomorphic, the mathematical reasoning that led the students to each representation was very different. The two isomorphic representations are Shoshanna's final representation, Figure 5, in the handshake context and Betsy' array with Shoshanna's crossing offs, Figure 9, in the set-theoretic context. These final

[5] By replacing each ordered pair in the set-theoretic representation (Figure 9) with the first element in each ordered pair and reordering the rows and columns, this isomorphism can be seen.

representations, in both contexts, were arrays built from the set of all possible handshakes or the set P×P, but the construction of each array was very different. In the handshake context, the students started with the global idea of assigning handshakes, but built up that global picture, the array, as they determined the local outcomes of the handshake assignments of each couple. In the set-theoretic context, they began with a global picture of every possible handshake at the party and then eliminated ordered pairs as they worked through the problem. After they had crossed out ordered pairs, the global picture had changed, allowing them to determine the local outcomes of that picture, namely the number of ordered pairs that end in each element.

Consider then the thinking of the students in each context. In the handshake context, the students' work was directed towards finding the number of handshakes of the hostess. They inherently understood two of the conditions, you cannot shake your own hand (compare to Condition 1 in the set-theoretic context) and if I shake your hand, you shake mine (compare to Condition 3 in the set-theoretic context). They were not held back trying to understand why no one can shake the same number of hands, excluding the host (compare to Condition 4). Thus, their familiarity with the context allowed them to use examples and then to extend their thinking by drawing connections between the conditions of the problem and the examples they chose to use. These same connections could not be drawn in the set-theoretic context until they had first developed meaning for each of those conditions. Even then, the group was not able to draw the same connections until they implemented those conditions on a representation that gave them access to each of those conditions, specifically the symmetry condition.

The students' reasoning in the handshake context can be described as using the relationships in the problem to find a key idea that could then be used as the basis of an argument for the solution. Representations were used and refined along the way until a representation was found that illustrated the argument that they had built for the solution. In contrast, the students' reasoning in the set-theoretic context can be described as using representations in order to build meaning from which they could begin to think about possible solutions. A final representation then became the foundation from which they were able to pose possible solutions and reason about the relationships in the problem. Once the solution fell out of their reasoning, they built an argument that was based on their representation and the structure of that representation.

These students did not appear to recognize any connections between the problem set in both contexts until Shoshanna began to explain her solution and the pattern she organized in the solution in the set-theoretic context. Perhaps it was not until they discovered the solution that they recognized similarities in their thinking that paralleled their thinking in the handshake context. Even though the students used the same example (the host and hostess both shake eight hands and the number of ordered pairs that end in m_1 are eight and the number of that end in f_1 are eight), they did not recognize the connection at that point.

Once the connection was made between the solutions of the problem in the two contexts, it was not obvious to the students that they were the same problem. They stated that the two problems were similar, but then questioned whether they were the same. This led them to look for direct relationships between the two problems. As they began to interpret each set and condition in the set-theoretic context in terms of the people and conditions in the handshake context, they were able to connect the pieces directly between the two contexts and then build an isomorphism. Only then did they decide that it was in fact the same problem.

Conclusions

Using the language of Freudenthal (1991), the two contexts examined here might be thought of as exemplifying a rich or a poor structure. Freudenthal refers to a rich context as one in which the structures need to be discovered. In such a context, there are more relationships available to exploit. The structures in a poor context are imposed upon one as a result of the minimal relationships that are available in that context. According to these descriptions of rich and poor contexts, it can be concluded that the handshake context is a rich context and the set-theoretic is a poor context. Indeed, the structures in the handshake context were discovered and built by the students as they discovered the various relationships that existed in that context, i.e. could the conditions of the problem affect the possible assignments of handshakes? The structures in the set-theory context were imposed upon the students more than they were discovered. Their representations matched the structure of the problem before they matched their own reasoning.

It is clear that both rich and poor contexts allow one to meet many different needs in mathematics and are valuable in those respective domains. Consider the remarkable development of warrants and key ideas in the handshake context as well as the impressive arrays constructed by these students in both contexts and the powerful reasoning built from those arrays. It is clear that although both contexts offered valuable mathematical experiences, in each context, those experiences were very different. Each context affected the purposes of the students and the choices that they made. Their arguments in each context were also influenced by those different purposes and choices as illustrated in their representations.

The argument in the handshake context was based on the warrant for the 8-0 relationship and the relationship between the other couples. In the set-theoretic context, the 8-0 idea and the argument built from that idea could not be considered until a representation was built that could offer them a new context from which that idea could be discussed. Consider the group's presentation of the solution in the set-theoretic context. Although they had already discovered the connection between the two contexts, they chose to explain their solution strictly in the set-theoretic context. Rather than referring to Condition 4 by discussing "the number of ordered pairs that end in a given element," as they had originally, they referred to Condition 4 by discussing the rows in the array. By using the context of the rows and later the columns of the array, they could offer an argument similar to the one used in the handshake context. Having invested considerable effort in constructing an argument in this context, perhaps these students wished to demonstrate that the argument could be carried out completely in the context they were given. They may have felt that in some way the problem was devalued by simply appealing to the isomorphic problem which everyone had solved much earlier.

Ellen Langer (1989) stated, "A context is a mindset, a premature cognitive commitment" (p. 37). The purposes and choices of these students illustrate their cognitive commitments in each context. In the handshake context the students' work pointed directly at solving the problem, building notation and representations that served that purpose. In the second context, their work centered on understanding the problem, interpreting the notation and building representations to gain that understanding. Indeed the rich and poor characteristics of each context affected those purposes and choices and thus affected what their cognitive commitments were. It was not until the students anchored their reasoning to a representation in the same way as the handshake context, allowing the cognitive aspects of their work to emerge, that they recognized that the two problems were in fact the same. This study provides an existence proof that the context of a problem may strongly affect lines of reasoning.

Implications for Teachers

Now that these lines of reasoning have been analyzed in both contexts, what do teachers gain from understanding the differences that arise in these lines of reasoning? In the handshake context, the students were strongly influenced by their own intuition and relied on a representation to anchor their reasoning in a way that helped them to overcome their intuition. The representations in the set-theoretic context were used to help the students anchor their reasoning about the meaning of the problem in a way that lessened the complexity of the notation. When teachers present problems to their students, similar difficulties may arise as a result of students' purposes that are elicited by the context of the problem. When teachers understand these difficulties, they can then help their students to overcome them, drawing upon representations and key ideas that will guide them to the type of purposeful thinking and reasoning that the teacher intended. Also, when teachers understand the types of purposeful thinking and reasoning that may arise from a particular context, they can make informed decisions about how to present the problems in a way that will allow their students to develop the reasoning that was intended.

The students' development of an isomorphism in this study is also significant for teachers. As the students compared the sets and conditions between the two contexts, their understanding of the set-theoretic context seemed to solidify and confirm their reasoning in that context. Similarly, as teachers offer opportunities for students to draw comparisons between problems in various contexts, be it "real world" contexts, graphs, tables, or algebraic notation, the students' understanding may be solidified.

Further Questions of Interest

The results of this study also contribute to further research in mathematics education. Because of the students' completely independent approach to the problem in each context, the following question arises: How might students' purposes, choices, and reasoning be affected if the two contexts are approached in the reverse order, first the set-theoretic context, then the handshake context? This question would help teachers to understand the importance of offering problems in one context before problems in a second context. Consider now the title of this study, "One Problem, Two Contexts". Additional research could advance the idea of "one problem". In what sense was the same problem offered to these students? In what sense can a problem be divorced from the same context? Finally, suppose that a problem was posed in a less artificially imposed setting, rather than a party with unrealistic restrictions on the handshakes, and a second isomorphic problem was created using formal mathematical notation. Would the same differences in students' purposes, choices, and reasoning arise between the two contexts?

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